

TABLES AND GRAPHS FOR FACILITATING THE COMPUTATION OF SPECTRAL ENERGY DISTRIBUTION BY PLANCK'S FORMULA

CONSISTING OF SEVEN SHEETS (FIVE CHARTS) AS FOLLOWS: TEXT (SHEET 1). GRAPHS: 1,000 TO 5,000° K. (SHEETS 2, 3, 4; CHARTS 1, 2, 3); 5,000 TO 8,000° K. (SHEET 5; CHART 4); 8,000 TO 24,000° K. (SHEET 6; CHART 5). TABLES (SHEET 7).

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ABSTRACT

Very frequently it has been necessary to compute the energy distribution of a "black body" in the visible spectral region, at some particular temperature, from Planck's formula:

$$E_{\lambda} = \frac{C_1 \lambda^{-5}}{e^{\frac{C_2}{\lambda \theta}} - 1}$$

Such computation consumes much time and labor; in consequence, the following short cuts in the way of tables and graphs have been devised.

Using the equation in the form

$$\frac{E_{\lambda}}{E_m} = \left(\frac{\lambda}{\lambda_m} \right)^5 \left(\frac{C_2}{e^{\frac{C_2}{\lambda \theta}} - 1} \right) \left(\frac{C_2}{e^{\frac{C_2}{\lambda_m \theta}} - 1} \right)^{-1}$$

a table has been made with values of $\lambda \theta$ and corresponding values of $\frac{E_{\lambda}}{E_m}$. For any given temperature the wave lengths in the visible, and partly in the ultra-violet and infra-red regions, corresponding to these values of relative energy, can readily be obtained from the products $\lambda \theta$. Hence the spectral energy curve can be plotted at once.

Two sets of tables have also been made up, each for 38 temperatures, in which the relative energy values computed directly from Planck's formula are recorded for the wave lengths 400 to 720 $m\mu$ at intervals of 10 millimicrons—one set having the energy value equal to 100 at 560 $m\mu$, the other equal to 100 at 560 $m\mu$. The temperature range is from 1,000 to 28,000° K. (accuracy, within 0.15 per cent).

From one of these tables a series of isochromatic curves has been constructed. The relative energy has been plotted (accuracy, within 0.33 per cent) against the absolute temperature, and one complete curve drawn for each of the 33 wave lengths. From these graphs, the energy distribution for the visible region can be obtained in a few minutes for any temperature whatever in the range above specified.

I. INTRODUCTION

In connection with the work of the color laboratory, very frequent use is made of the spectral energy distribution at various temperatures of "black body" radiation, as represented by Planck's formula:

$$E_{\lambda} = \frac{C_1 \lambda^{-5}}{e^{\frac{C_2}{\lambda \theta}} - 1} \quad (1)$$

E_{λ} is the radiant energy for any wave length λ , at any temperature θ (degrees absolute); C_1 and C_2 are constants, and e is the base of the natural logarithms. As a rule it is desired to know the relative values of E at a given temperature for a range of 320 $m\mu$ throughout the visible spectrum; that is, 400 to 720 $m\mu$, at intervals of 10 $m\mu$. Moreover, these values must conform to a scale such that at some particular wave length E will always be the same for all temperatures, for instance, equal to 100 at wave length 560 $m\mu$.² When computed directly from the formula, this proves to be a long and tedious process; therefore, certain short cuts have been devised. Tables have been made and curves plotted which greatly facilitate this work. It was thought worth while to publish these for the benefit of others who make similar use of this formula. The following reference tables have been found of great assistance: F. W. Newman and J. W. D. Glaisher, Tables of Exponential Functions; C. E. VanOrstrand, Tables of the Exponential Function.³

II. EXPLANATION OF TABLES

1. TABLE 1

Instead of using Planck's equation in the form given above, it was combined⁴ with Wien's displacement law connecting the temperature with the wave length at which the maximum energy occurs; that is,

$$\lambda_m \theta = A, \text{ a constant} \quad (2)$$

This maximum energy is given by

$$E_m = \frac{C_1 \lambda_m^{-5}}{e^{\frac{C_2}{\lambda_m \theta}} - 1} \quad (3)$$

Substituting from (2) $\lambda_m = \frac{A}{\theta}$, equation (3) takes the form

$$E_m = \frac{C_1 \left(\frac{A}{\theta} \right)^{-5}}{e^{\frac{C_2}{\frac{A}{\theta} \theta}} - 1} = \frac{C_1 A^{-5}}{e^{\frac{C_2}{A}} - 1} \quad (4)$$

Dividing (1) by (4) we obtain

$$\frac{E_{\lambda}}{E_m} = \left(\frac{\lambda}{\lambda_m} \right)^5 \left(\frac{C_2}{e^{\frac{C_2}{\lambda \theta}} - 1} \right) \left(\frac{C_2}{e^{\frac{C_2}{\lambda_m \theta}} - 1} \right)^{-1} \quad (5)$$

An inspection of this equation will show that the right-hand side is a function of $\lambda \theta$ alone. The values of the constants used in the computations are:

$$A = 2,890 \text{ micron degrees} \\ C_2 = 14,350 \text{ micron degrees}^2$$

Table 1 has been arranged with a wide range of values of $\lambda \theta$ and corresponding values of $\frac{E_{\lambda}}{E_m}$, computed to four significant figures.

Use of Table 1.—From the products $\lambda \theta$ the wave lengths can be readily computed for any temperature θ . (These wave lengths are in microns, and must be multiplied by 1,000 to be expressed in millimicrons.) The values of relative energy at these wave lengths are already known from the table; hence the distribution curve for this temperature can at once be plotted. The next step is to transform the values $\frac{E_{\lambda}}{E_m}$ so that at 560 $m\mu$ the energy will be equal to 100. This is accomplished by reading from the curve the energy coordinate for 560 $m\mu$, taking its reciprocal (times 100) and multiplying the other coordinates by this factor. The spectral energy curve is thus well defined and will pass through the required point at 560 $m\mu$.

It is frequently necessary to know the energy at every 10 $m\mu$ beginning at 400 $m\mu$. These values, while not obtainable directly from the $\lambda \theta$ computations, are readily obtained from the distribution curve mentioned above, and can be read with accuracy (about 0.2 to 0.6 per cent), provided the scale has been suitably chosen.

In order, therefore, to obtain energy values for any temperature θ at every 10 $m\mu$, with a given value at 560 $m\mu$, the use of Table 1 involves four distinct steps: (1) Computation of λ from the products $\lambda \theta$; (2) plotting of λ against $\frac{E_{\lambda}}{E_m}$ given in the table; (3) reading from this curve the values of $\frac{E_{\lambda}}{E_m}$ at intervals of 10 $m\mu$; and (4) multiplying these values by the reciprocal of the value at 560 $m\mu$, and the product by 100.

This process of computation, while more easily and somewhat more rapidly performed than direct calculation from Planck's formula, still consumes a considerable amount of time. Moreover, without extreme care in the plotting of the curves, the accuracy obtainable may not be greater than 0.6 per cent. The use of Table 1 can not be recommended for great speed or great accuracy, under the conditions imposed in the introduction; but it is very useful to give readily the shape of the energy distribution curve and the relative values of the energy when the above conditions need not be met. Moreover, with the use of Table 1, the curve can be extended into the ultra-violet and infra-red regions of the spectrum.

These tables have been prepared directly from Wien's and Planck's formulas, with the exception only of the values for 1,000° K., which were calculated directly from Table 1. Wien's formula was used for temperatures 1,100 to 2,000° K., Planck's formula for 2,200 to 28,000° K. Since the energy values given in these tables are all relative to the value chosen for wave length 560 (or 590) $m\mu$ the constant C_1 does not enter into consideration. C_2 is taken equal to 14,350 micron degrees. The values in these tables are considered correct to within 0.15 per cent.⁵

The temperatures for which the energy values are given in these tables are as follows:

2. TABLES 2A AND 2B

1,000 to 2,000, by steps of 100°.
2,000 to 3,000, by steps of 200°.
3,000 to 4,000, by steps of 250°.
4,000 to 7,000, by steps of 500°.
7,000 to 10,000, by steps of 1,000°.
10,000 to 28,000, by steps of 2,000°.

In Table 2A the energy has been made equal to 100 at 590 $m\mu$ for each temperature; in Table 2B the value 100 has been chosen at 560 $m\mu$.

III. EXPLANATION OF GRAPHS

The plotting of the graphs was suggested by the existence of Tables 2A and 2B. The energy distribution is often required to be known for some temperature intermediate to those given, and a series of curves based on these tables should be very suitable to yield these data. This led to a series of isochromatic curves, one for each wave length at intervals of 10 $m\mu$, from 400 to 720 $m\mu$. E_{λ} can be read directly for any temperature desired within the range 1,000 to 28,000° K.

The plotting of these curves met with some difficulties for the following reasons: (1) It was desired to obtain an accuracy of plotting and of reading to 0.33 per cent. (2) The energy values vary from 0.007 (approximately) to 3,000; in other words, the largest value is about 400,000 times the smallest value. This necessitates several changes of scale for the energy coordinates.⁷ (3) The isochromatics crowd very closely together between temperatures 4,000 to 7,000°.

In view of these considerations, it has been found necessary to plot the curves in sections on 40 by 50 cm sheets. Only that series is given for which E_{λ} has the value of 100 at 560 $m\mu$. The other scale of values (100 at 590) can readily be obtained from these by the use of a suitable factor. The accuracy mentioned

¹ Cf. Table given in Jour. Op. Soc. Am., 4, p. 317, 1920; W. E. Forsythe. This table is convertible into Table 2A or 2B by the use of suitable factors. Some of the values, however, have been found to be slightly in error, and it is understood are in process of revision.

² The logarithmic scale was discarded as unsuitable for certain regions. Nomograms proved rather cumbersome because of the form the equation takes for transforming the energy values to 100 at 560 $m\mu$ etc.; that is, $\frac{E_{\lambda}}{E_m} = f(\theta) \phi(\lambda \theta)$.

above (0.33 per cent) should apply except for E_{λ} less than 3.5 and possibly also on very steep parts or near the line $\theta = 1,000$ ° K. The temperature can be estimated to within 1° for the range 1,000 to 8,000°, and to within 5° for the range 8,000 to 28,000°. The curves have been tested by comparing values furnished by the graphs (for given temperatures other than those recorded in Table 2B) with the values computed for these temperatures.

IV. PRACTICAL APPLICATION

It has been shown⁸ that a number of the ordinary illuminants have a spectral energy distribution that approximates closely that of a Planckian radiator. This property is valuable in that the unknown spectral energy distribution of an illuminant can very readily be determined if it can be matched in color with a so-called "black body." The temperature to which the "black body" is brought in order to produce color match with the illuminant is defined as the color temperature of the latter. Since this quantity is not difficult to determine in laboratories having suitable equipment, the energy distribution in the visible region of such illuminant can at once be recorded by reading from the charts the energy values for the given temperature.

V. SUMMARY

If it is required to know the spectral distribution of a Planckian radiator at any temperature θ throughout the visible and into the ultra-violet and infra-red regions, Table 1 affords a ready means of computing it with very little labor. But if the distribution curve is required to pass through a certain point (for instance, 100 at 560 $m\mu$), and if the values of energy are desired for 10 $m\mu$ intervals, further computation is necessary. Tables 2A and 2B have been computed giving these values for 38 temperatures in the interval 1,000 to 28,000° K.

In order to obtain directly the spectral distribution for any temperature whatever in this interval, at wave lengths 10 $m\mu$ apart, and having the value 100 at 560 $m\mu$, a series of isochromatic curves has been constructed. From these curves, which are plotted to a large scale, the energy values can be read to an accuracy usually within 0.33 per cent, the temperature values within 0.1 per cent.

VI. APPENDIX

The question may arise as to what variation will be caused in the relative values of E_{λ} by the adoption of a value of C_2 different from 14,350; for example, Coblentz's most recent value¹⁰ for this constant is 14,320; the International Critical Tables has adopted 14,330; and further research may eventually cause further, though probably very small, revision of these latest values.

An inspection of the equation, $E_{\lambda} = \frac{C_1 \lambda^{-5}}{e^{\frac{C_2}{\lambda \theta}} - 1}$ shows that the

effect of a variation of C_2 with θ constant is the same as, but opposite in sense to, a variation of θ with C_2 constant.

So far as the graphs are concerned, in which values of E_{λ} are plotted against values of θ for the various wave lengths, we have, therefore, a ready means of obtaining true values of E_{λ} (within the precision already noted in the main part of the paper) for any value of C_2 which it may be desired to use. The method is simply this: Compute the percentage difference between 14,350 and the value to be used. Call this percentage difference p . Multiply the temperature by $1 \pm p/100$ and read values of E_{λ} from the graphs at this corrected temperature. If C_2 is decreased, θ must be increased, and vice versa.

While true values of E_{λ} may thus be obtained for any value of C_2 , and as readily as for the value 14,350, the difference in E_{λ} obtained by using 14,320 or 14,330 is so small as to be negligible for most work for which these tables and graphs will be used. In the first place, the difference in temperature necessary to make the correction on the graphs is about 0.2 per cent, the order of magnitude of differences in color temperature determinations by experienced observers. In the second place, the differences in values of E_{λ} relative to that at 560 $m\mu$ caused by this 0.2 per cent change in C_2 is but a few tenths of 1 per cent, except at the extreme wave lengths for temperatures below about 1,500°, where they may become greater than 1 per cent. Spectral energy measurements can scarcely be guaranteed within 1 per cent, while the discrepancies between spectral energy measurements and values of energy computed for the corresponding color temperature may be several per cent in the end regions of the visible spectrum.

While, therefore, true values of E_{λ} may be readily obtained from the graphs for any value of C_2 , it is probable that for most work the error involved in using values of E_{λ} based on $C_2 = 14,350$ —that is, those given by the tables or read from the graphs without the temperature correction—will be negligible.

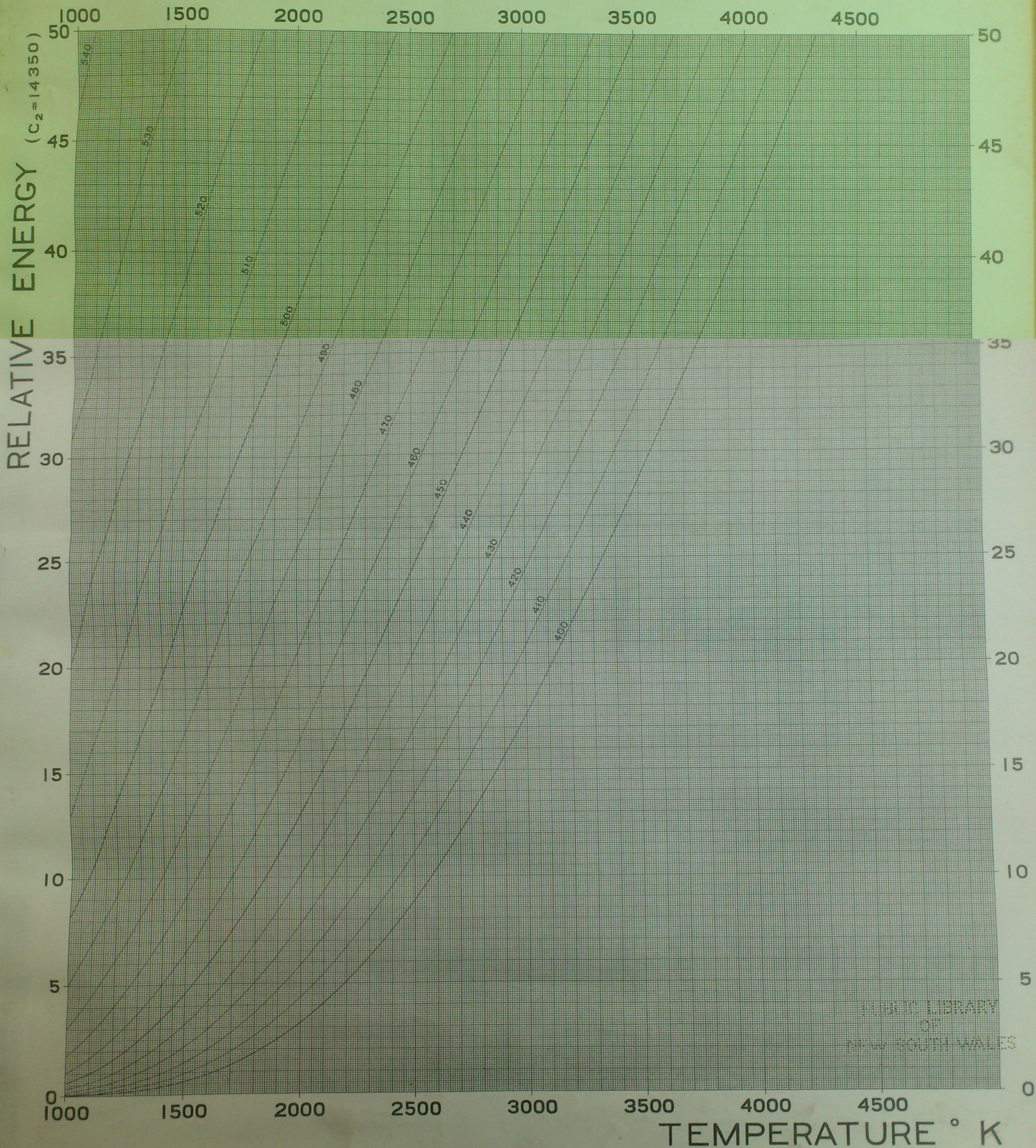
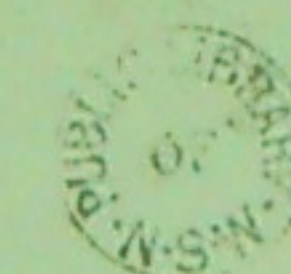
⁸ Hyde and Forsythe, The quality of light from an illuminant, as indicated by its color temperature, J. Frank. Inst., 188, pp. 353-354, 1917. Priest, Measurement of the Color Temperature of Illuminants, Artificial Light Sources by the Method of Rotatory Dispersion, U. S. Bureau of Standards, 1920.

⁹ The authors wish to acknowledge the suggestions of Dr. K. S. Gibson in connection with the appendix.

¹⁰ B. S. Sci. Paper No. 426, December, 1922.

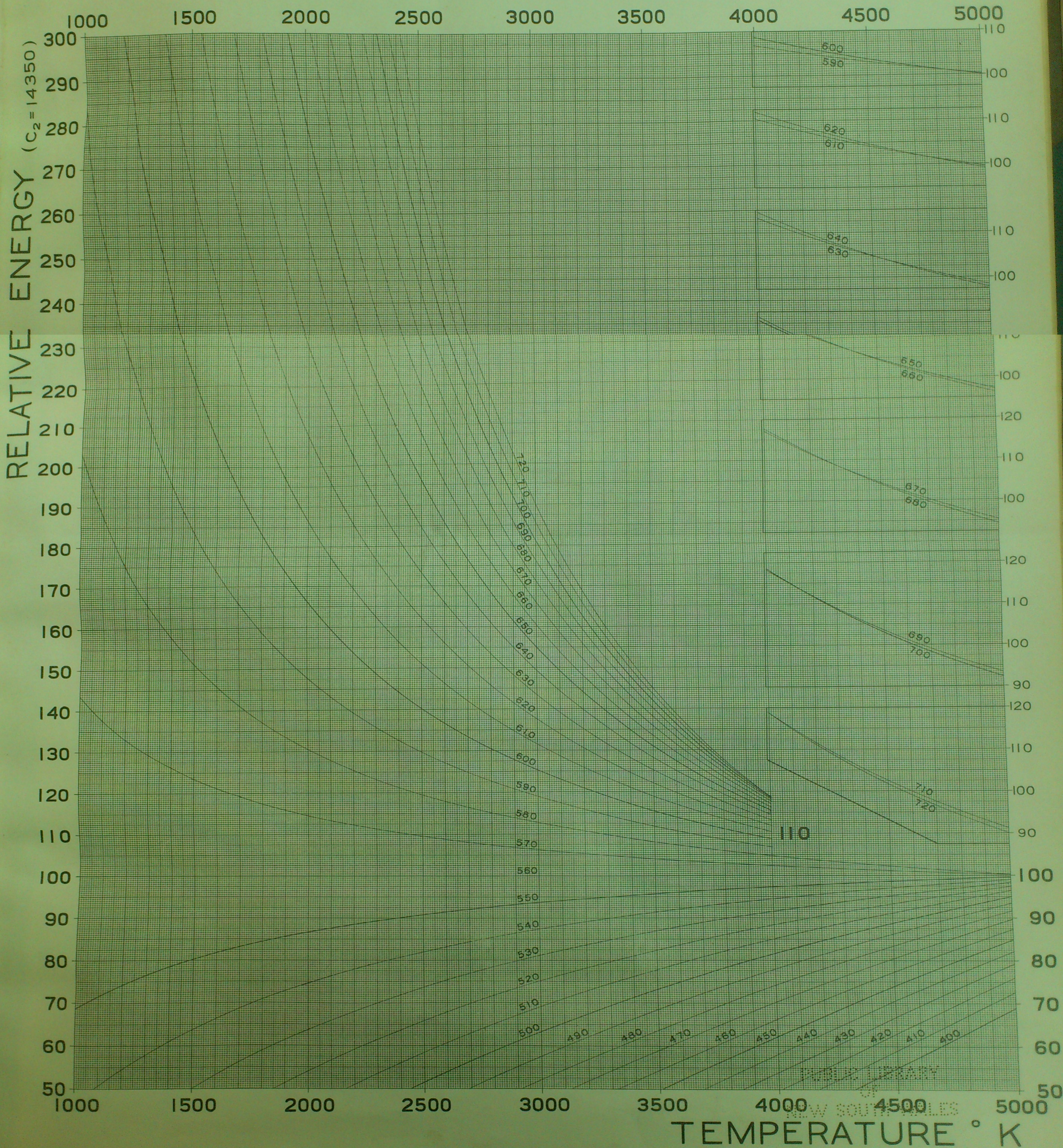
ENERGY DISTRIBUTION BY PLANCK'S FORMULA

100 AT WAVE LENGTH 560 mμ



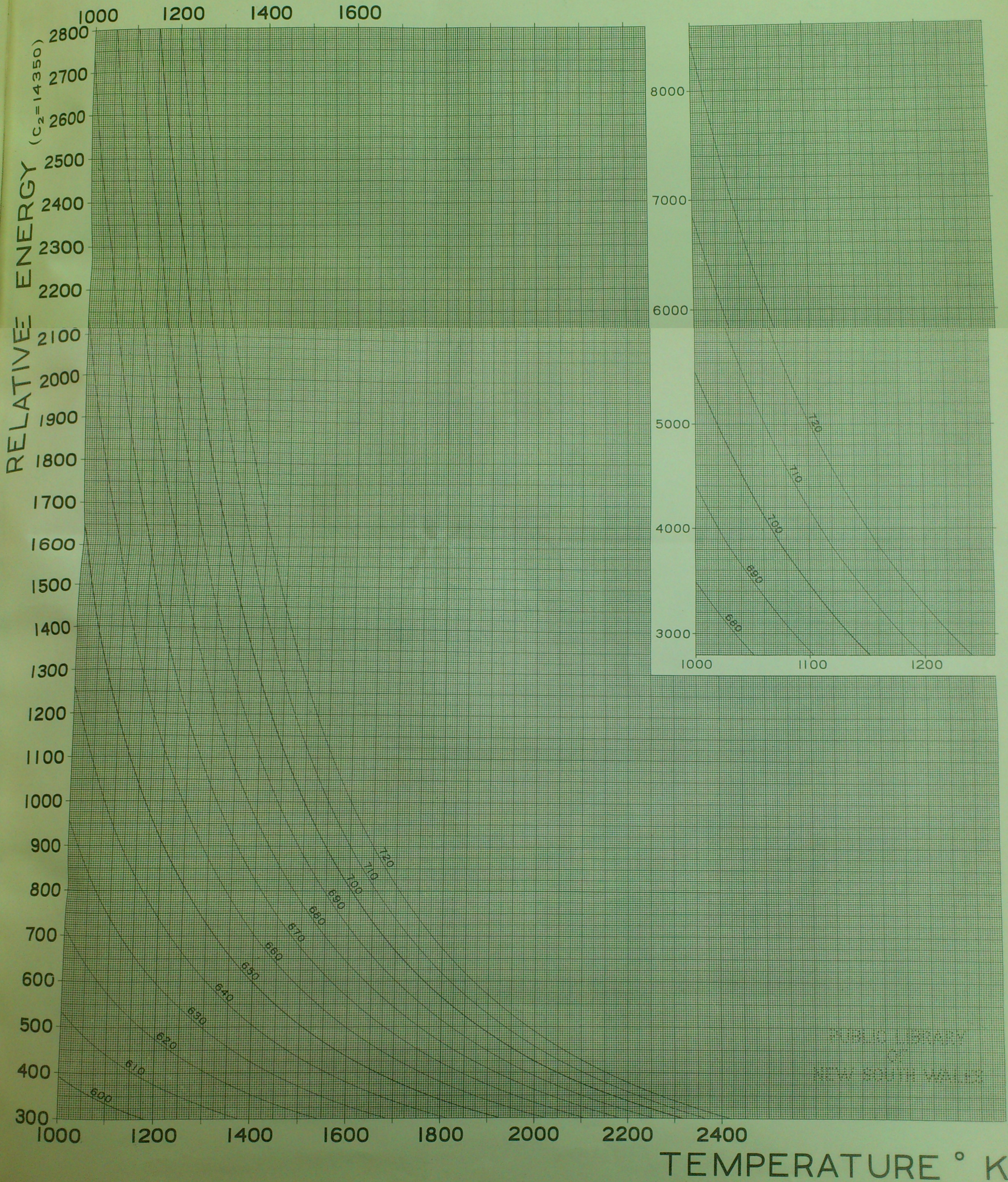
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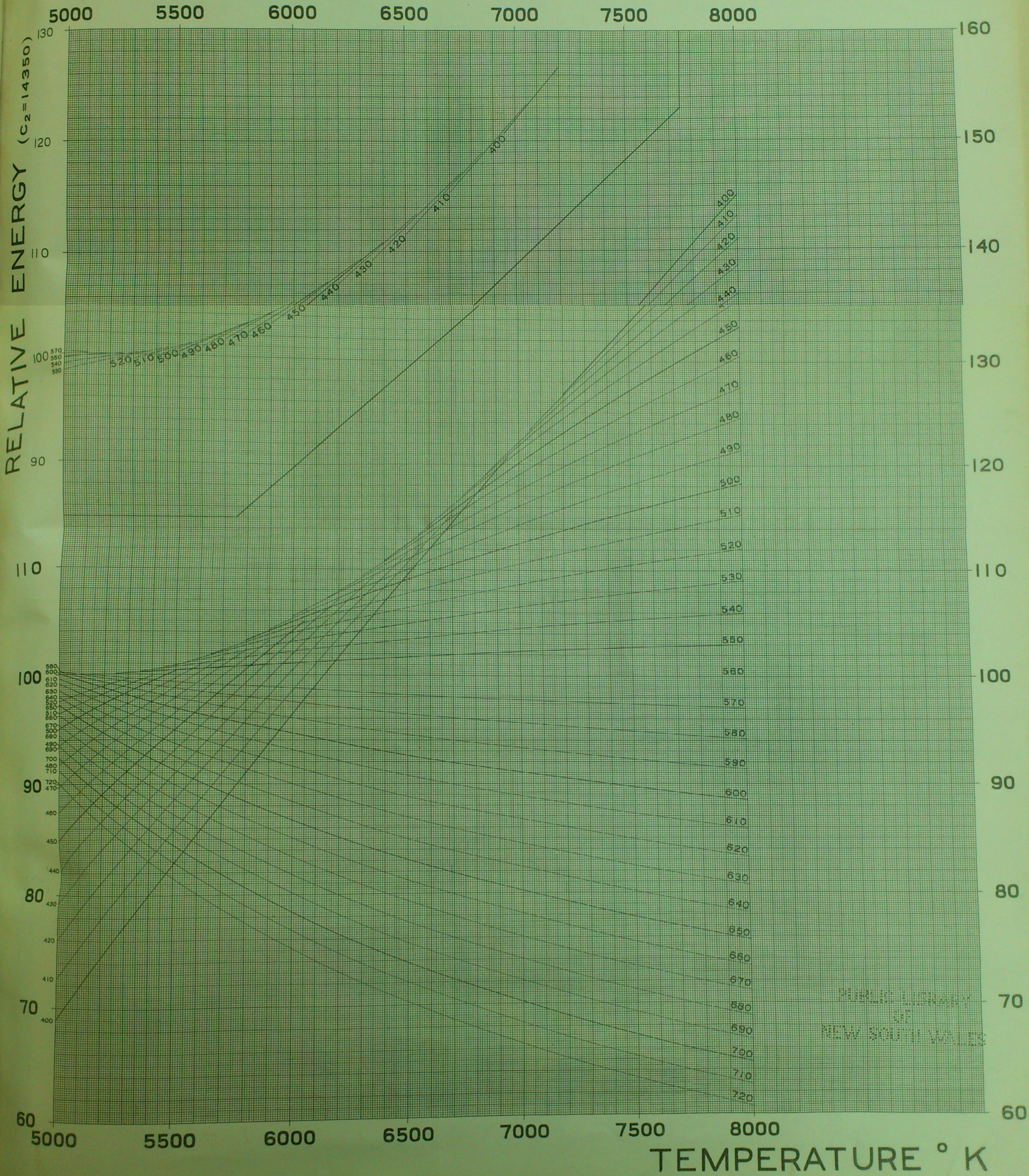
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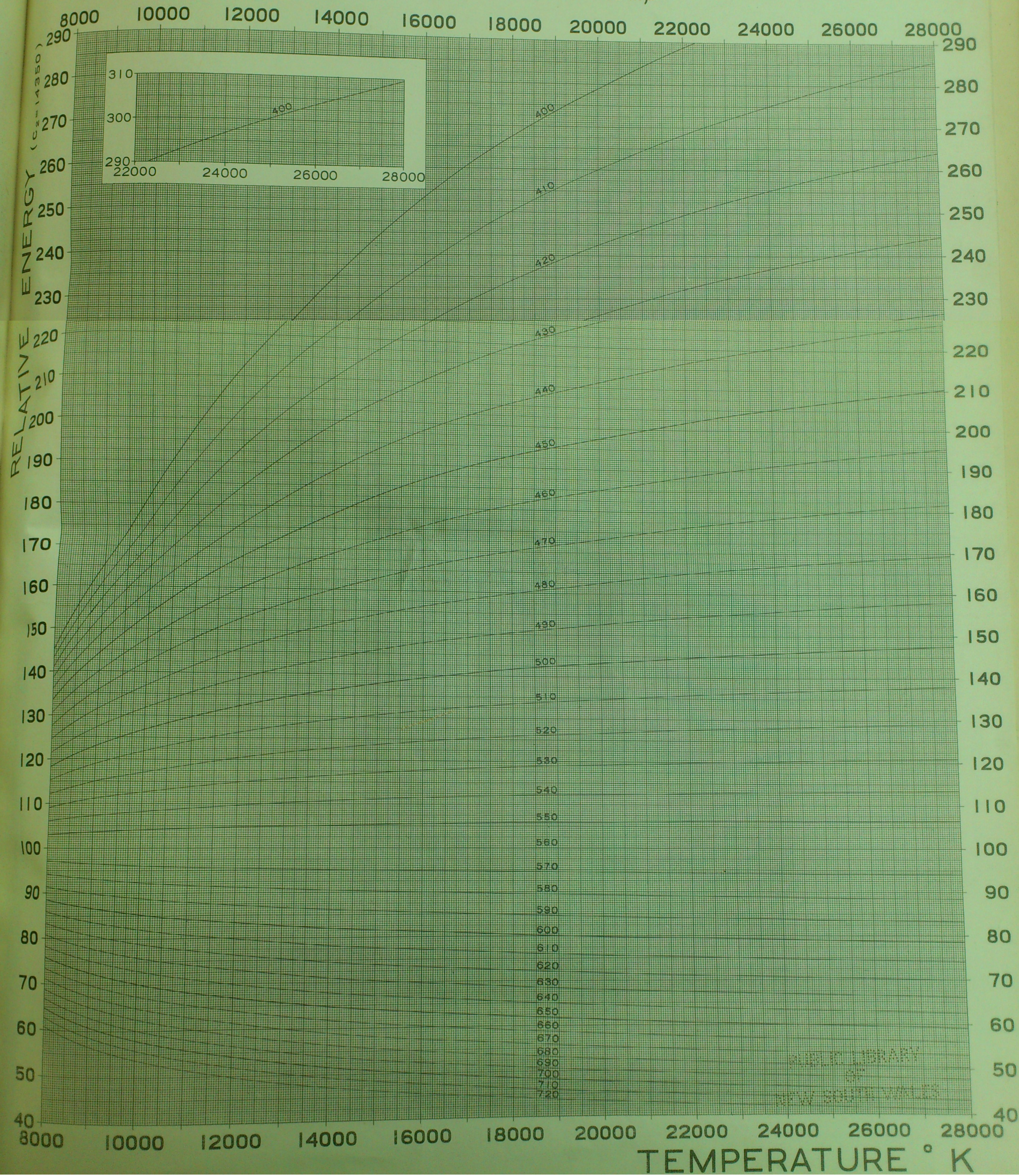
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TABLES AND GRAPHS FOR FACILITATING THE COMPUTATION OF
SPECTRAL ENERGY DISTRIBUTION BY PLANCK'S FORMULATABLE 1. $\frac{E_{\lambda}}{E_m} = \left(\frac{A}{\lambda} \right)^5 \left(\frac{C_2}{\lambda^5} - 1 \right) \left(\frac{C_1}{\lambda^5} - 1 \right)^{-1}$ [A = 2,890 micron degrees, $C_1 = 14,350$ micron degrees, $C_2 = 142.37$]

λ in micron degrees	$\frac{E_{\lambda}}{E_m}$	λ in micron degrees	$\frac{E_{\lambda}}{E_m}$	λ in micron degrees	$\frac{E_{\lambda}}{E_m}$
400	0.00000007367	1,180	0.06562	4,700	0.6200
410	0.0000001562	1,200	0.07991	4,800	0.5997
420	0.0000003185	1,220	0.08778	4,900	0.5740
430	0.0000006670	1,250	0.09720	5,000	0.5521
440	0.000001193	1,300	0.1343	5,200	0.5103
450	0.000002201	1,350	0.1548	5,400	0.4715
460	0.000004000	1,400	0.1887	5,600	0.4355
470	0.000007589	1,450	0.2254	5,800	0.4023
480	0.000014171	1,500	0.2607	6,000	0.3716
490	0.000027192	1,550	0.3059	6,200	0.3435
500	0.00005154	1,600	0.3486	6,400	0.3177
510	0.0001005016	1,650	0.3922	6,600	0.2940
520	0.000207818	1,700	0.4363	6,800	0.2723
530	0.00041196	1,750	0.4804	7,000	0.2523
540	0.00071799	1,800	0.5241	7,200	0.2340
550	0.0013660	1,850	0.5669	7,400	0.2173
560	0.0025561	1,900	0.6086	7,600	0.2019
570	0.0045561	1,950	0.6487	7,800	0.1878
580	0.0085561	2,000	0.6871	8,000	0.1748
590	0.0155561	2,050	0.7236	8,200	0.1629
600	0.0275561	2,100	0.7579	8,400	0.1519
610	0.0455561	2,150	0.7900	8,600	0.1414
620	0.0735561	2,200	0.8196	8,800	0.1314
630	0.0003706	2,250	0.8470	9,000	0.0970
640	0.0001889	2,300	0.8715	9,200	0.07932
650	0.0006389	2,350	0.8945	10,400	0.07036
660	0.0012663	2,400	0.9167	11,200	0.06261
670	0.0002781	2,450	0.9325	11,600	0.05588
680	0.0003706	2,500	0.9479	12,000	0.05001
690	0.0001889	2,550	0.9622	12,400	0.04468
700	0.002135	2,600	0.9722	12,800	0.04029
710	0.002655	2,650	0.9815	13,200	0.03644
720	0.003276	2,700	0.9888	13,600	0.03325
730	0.004597	2,750	0.9950	14,000	0.02986
740	0.007136	2,800	0.9974	14,400	0.02713
750					
760	0.01018	2,850	0.9995	14,800	0.02469
770	0.01018	2,850		15,200	0.02243
780	0.01419	2,900	0.9995	15,600	0.02069
790	0.01944	2,950	1.0000	16,000	0.01925
800	0.02615	3,000	0.9990	16,400	0.01809
810	0.03458	3,050	0.9967	16,800	0.01718
820	0.04503	3,100	0.9886	17,200	0.01645
830	0.05710	3,200	0.9762	18,000	0.01525
840	0.07185	3,300	0.9603	17,500	0.01478
850	0.08736	3,400	0.9408	18,500	0.01416
860	0.09169	3,500	0.9209	19,000	0.01346
870		3,600	0.8982	19,500	0.01280
880	0.11134	3,700	0.8742	19,000	0.01207
890	0.1388	3,800	0.8472	19,500	0.01162
900	0.1682	3,900	0.8176	20,000	0.01120
910	0.20034	4,000	0.7856	20,500	0.01084
920	0.23609	4,100	0.7515	21,000	0.01050
930		4,200	0.7144		
940	0.0314	4,300	0.7453		
950	0.03852	4,400	0.7193		
960	0.04441	4,500	0.6938		
970	0.05436	4,600	0.6686		
980	0.05794	4,700	0.6440		